AN INTRODUCTION TO THE LATTICE OF TORSION CLASSES

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Abstract. In this expository note, I present some of the key features of the lattice of torsion classes of a finite-dimensional algebra, focussing in particular on its complete semidistributivity and consequences thereof. This is intended to serve as an introduction to recent work by Barnard–Carroll–Zhu and Demonet–Iyama–Reading–Reiten–Thomas.

Let A be a finite-dimensional algebra over a field k. We write $\mathsf{mod}\,A$ for the category of finite-dimensional left A-modules. There is a class of subcategories of $\mathsf{mod}\,A$ which are called torsion classes. The torsion classes, ordered by inclusion, form a poset which we denote $\mathsf{tors}\,A$. This poset is in fact a lattice, and its lattice-theoretic properties have recently been the focus of some attention, as in [Ja, GM, BCZ, DI+, AP].

In this note I will present some of the interesting features of these lattices. The proofs in this note are self-contained except for the final section, where we present without proof an application of these ideas to the study of finite semidistributive lattices from [RST]. This note is intended as a gentle introduction to the subject. No results in this note are new. The presentation is, of course, novel in some respects, and I hope that it is helpful as an introduction to the subject.

Let me now quickly summarize the contents of this note. Terms which are undefined here will be introduced later where they logically fit. In addition to presenting the easy explanation that $\mathsf{tors}\,A$ is a lattice, I will prove the result of Barnard, Carroll, and Zhu [BCZ] that the completely join irreducible elements of $\mathsf{tors}\,A$ are in bijection with the bricks of A. I will show that $\mathsf{tors}\,A$ is completely semidistributive. I will not take the most direct route to this result, but rather spend some time developing independently properties of $\mathsf{tors}\,A$ and corresponding properties of semidistributive lattices, in an attempt to illuminate how semidistributivity gives us a helpful perspective through which to view the combinatorics of $\mathsf{tors}\,A$. I will show that $\mathsf{tors}\,A$ is weakly atomic. We will then see how an algebra quotient induces a lattice quotient map between the corresponding lattices of torsion classes, and study this lattice quotient. In the final section, I will present (without proof) a construction of finite semidistributive lattices developed in [RST], and inspired by the study of lattices of torsion classes.

1. Definition of torsion classes

For the elementary material in this section and the two following, a further reference is [ASS, Chapter VI].

A torsion class in $\mathsf{mod}\,A$ is a subcategory $\mathcal{T}$ of $\mathsf{mod}\,A$ which is

- closed under quotients (i.e., $Y \in \mathcal{T}$ and $Y \twoheadrightarrow Z$ implies $Z \in \mathcal{T}$).

2010 *Mathematics Subject Classification.* Primary: 16G20, Secondary: 16S90, 06B99.
Key words and phrases. Torsion classes, semidistributive lattices, lattice congruences.

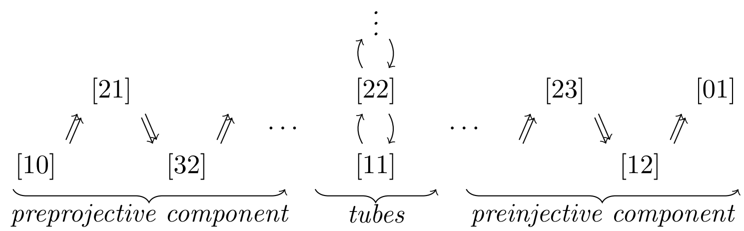


Figure 1. The AR quiver of the path algebra of the Kronecker quiver.

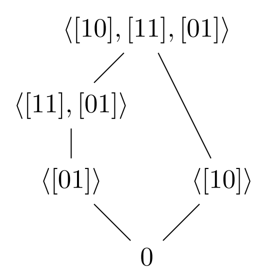
- closed under extensions (i.e., $X, Z \in \mathcal{T}$ and $0 \to X \to Y \to Z \to 0$ implies $Y \in \mathcal{T}$.

I should clarify that for me a subcategory is always full, closed under direct sums, direct summands, and isomorphisms. In other words, a subcategory of $\mathsf{mod}\, A$ can be specified as the direct sums of copies of some subset of the indecomposable modules of A.

We write $\mathsf{tors}\, A$ for the set of torsion classes of $\mathsf{mod}\, A$, and we think of it as a poset ordered by inclusion.

Example 1.1 (Type A_2)**.** *Our quiver Q is $1 \leftarrow 2$, and the algebra is the path algebra $A = kQ$. The category $\mathsf{mod}\, A$ has three indecomposable objects S_1, P_2, S_2, which I denote by their dimension vectors as $[10]$, $[11]$, and $[01]$, respectively.*

The torsion classes are as follows, where the angle brackets denote additive hull.



Example 1.2 (Type A_n)**.** *For an example in type A_n, where Q is $1 \leftarrow \cdots \leftarrow n$, see* [Kr].

Example 1.3 (Kronecker quiver)**.** *Let k be algebraically closed. Let Q be the quiver $1 \Longleftarrow 2$ and let $A = kQ$.*

The AR quiver is displayed in Figure 1, where I write $[ab]$ for an indecomposable module with dimension vector (a, b).

The tubes are indexed by points in $\mathbb{P}^1(k) = k \bigcup \{\infty\}$; they each look the same. The torsion classes consist of the additive hull of each of the following sets:

- *any final part of the preinjective component,*
- *all preinjectives and a subset of the tubes,*
- *all preinjectives, all tubes, and a final part of the preprojectives,*
- $S_1 = [10]$.

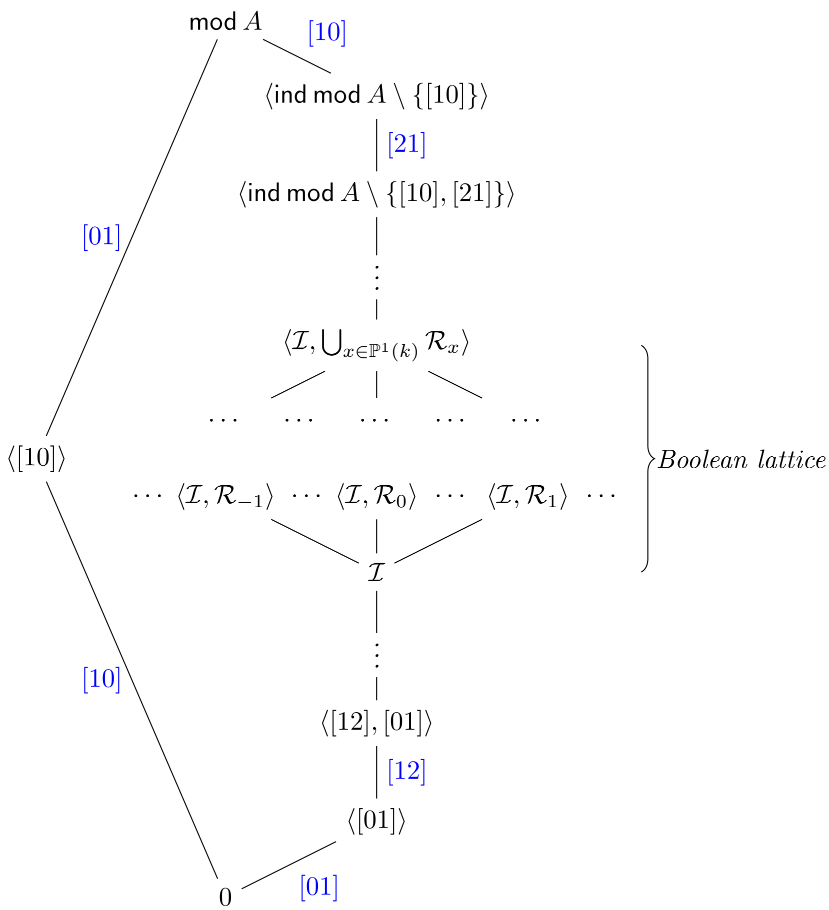


FIGURE 2. The lattice of torsion classes for A the path algebra of the Kronecker quiver. The labels on the edges should be ignored for now; they are the brick labelling $\hat{\gamma}$ defined in Section 11.

An image of the lattice of torsion classes is displayed in Figure 2. There, $\mathcal{I}$ denotes the preinjective component, and $\mathcal{R}_x$ denote the tube corresponding to $x \in \mathbb{P}^1(k)$. I write ind mod A *for the set of indecomposable A-modules.*

2. Specifying a torsion class

In general, how can we specify a torsion class? For $\mathcal{C}$ a subcategory of mod A, define $T(\mathcal{C})$ to be the subcategory whose modules are filtered by quotients of objects from $\mathcal{C}$. That is to say $M \in T(\mathcal{C})$ if and only if M admits a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with M_i/M_{i-1} a quotient of an object of $\mathcal{C}$ for all i.

Proposition 2.1. *For $\mathcal{C}$ an arbitrary subcategory, $T(\mathcal{C})$ is the smallest torsion class containing all the objects from $\mathcal{C}$.*

Proof. Suppose that $M \in T(\mathcal{C})$, so that we have a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with M_i/M_{i-1} a quotient of an object of $\mathcal{C}$ for all i. Consider some quotient of M, say $N = M/L$. Then define $N_i = (M_i + L)/L$, which forms a filtration of N. We see that N_i/N_{i-1} is a quotient of M_i/M_{i-1}, and therefore a quotient of an object of $\mathcal{C}$. This shows that $N \in T(\mathcal{C})$.

Suppose next that we have two modules M and N, both in $T(\mathcal{C})$, and an extension

$$0 \to M \to E \to N \to 0.$$

Now E has a two-step filtration $0 \subset M \subset E$, with $E/M \simeq N$, and we can refine the two steps of the filtration to filtrations of M and N with subquotients being quotients of $\mathcal{C}$, since we know such filtrations exist. This shows that $E \in T(\mathcal{C})$. It follows that $T(\mathcal{C})$ satisfies the two defining properties, and is therefore a torsion class.

$T(\mathcal{C})$ is the smallest torsion class containing $\mathcal{C}$ because any element of $T(\mathcal{C})$ is an iterated extension of quotients of $\mathcal{C}$, which must be in any torsion class containing $\mathcal{C}$. □

We now consider a second way to specify a torsion class. For $\mathcal{C}$ a subcategory of $\mathsf{mod}\, A$, define

$$^{\perp}\mathcal{C} = \{X \in \mathsf{mod}\, A \mid \mathrm{Hom}(X,Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Proposition 2.2. *For $\mathcal{C}$ an arbitrary subcategory, $^{\perp}\mathcal{C}$ is a torsion class.*

Proof. Let $M \in {}^{\perp}\mathcal{C}$. Let N be a quotient of M. Since there are no non-zero morphisms from M into any object of $\mathcal{C}$, the same holds for N, so $N \in {}^{\perp}\mathcal{C}$.

Suppose now that we have M and N in $^{\perp}\mathcal{C}$, and an extension:

$$0 \to M \to E \to N \to 0.$$

For any $Y \in \mathcal{C}$, we have $\mathrm{Hom}(M,Y) = 0$ and $\mathrm{Hom}(N,Y) = 0$, and it follows from the left exactness of the Hom functor that $\mathrm{Hom}(E,Y) = 0$ as well. We deduce that $E \in {}^{\perp}\mathcal{C}$.

$^{\perp}\mathcal{C}$ satisfies the two defining conditions, and is therefore a torsion class. □

3. Torsion classes and torsion free classes

There is a dual notion to that of torsion class, namely that of torsion free class. A torsion free class in $\mathsf{mod}\, A$ is a subcategory closed under submodules and extensions. We write $\mathsf{tf}\, A$ for the torsion free classes of A, and we think of it as a poset ordered by inclusion.

As one should expect, in the setting of finite-dimensional algebras in which we work, the theory of torsion free classes is completely parallel to the theory of torsion classes. For $\mathcal{C}$ a subcategory of $\mathsf{mod}\, A$, define $F(\mathcal{C})$ to tbe the subcategory of $\mathsf{mod}\, A$ consisting of all modules filtered by submodules of modules from $\mathcal{C}$. Then $F(\mathcal{C})$ is the smallest torsion free class containing $\mathcal{C}$. We can also define

$$\mathcal{C}^{\perp} = \{Y \in \mathsf{mod}\, A \mid \mathrm{Hom}(X,Y) = 0 \text{ for all } X \in \mathcal{C}\}.$$

One easily checks that for any subcategory $\mathcal{C}$, the subcategory $\mathcal{C}^{\perp}$ is a torsion free class.

Proposition 3.1. *Let $\mathcal{T}$ be a torsion class, and let $X \in \mathsf{mod}\, A$. There is a maximum submodule of X contained in $\mathcal{T}$.*

Proof. If M and N are submodules of X, then we have a short exact sequence

$$0 \to M \to M + N \to N/(N \cap M) \to 0$$

If N and M are both in $\mathcal{T}$, it follows that $M + N$ is also. Because X is finite-dimensional by assumption, it therefore has a maximum submodule contained in $\mathcal{T}$. □

We denote this maximum submodule by $t_{\mathcal{T}}X$.

Proposition 3.2. *$X/t_{\mathcal{T}}X$ lies in $\mathcal{T}^{\perp}$.*

Proof. Suppose there were a non-zero map f from some $M \in \mathcal{T}$ to $X/t_{\mathcal{T}}X$. Then $\mathsf{im} f$ is a quotient of M, and therefore itself in $\mathcal{T}$. The preimage of $\mathsf{im} f$ in X is then an extension of $\mathsf{im} f$ by $t_{\mathcal{T}}X$, and is therefore also in $\mathcal{T}$, contradicting the definition of $t_{\mathcal{T}}X$. □

For any X in $\mathsf{mod}\, A$, we now have a short exact sequence:

$$(*) \qquad 0 \to t_{\mathcal{T}}X \to X \to X/t_{\mathcal{T}}X \to 0.$$

with the lefthand term in $\mathcal{T}$ and the righthand term in $\mathcal{T}^{\perp}$.

Proposition 3.3. *For $X \in \mathsf{mod}\, A$, any short exact sequence of the form*

$$0 \to X' \to X \to X'' \to 0$$

with X' in $\mathcal{T}$ and $X'' \in \mathcal{T}^{\perp}$ is isomorphic to $(*)$.

Proof. Viewing X' as a submodule of X, it must be contained in $t_{\mathcal{T}}X$. If the containment were strict, then X'' would not lie in $\mathcal{T}^{\perp}$. The result follows. □

We can now prove the following theorem:

Theorem 3.1. *The map $\mathcal{T} \mapsto \mathcal{T}^{\perp}$ is an inclusion-reversing bijection from torsion classes to torsion free classes. Its inverse is given by the map $\mathcal{F} \mapsto {}^{\perp}\mathcal{F}$.*

Proof. We already pointed out that $\mathcal{T}^{\perp}$ is torsion free. It is easy to see that ${}^{\perp}(\mathcal{T}^{\perp}) \supseteq \mathcal{T}$. For the other inclusion, suppose $X \in {}^{\perp}(\mathcal{T}^{\perp})$. Since $X/t_{\mathcal{T}}X \in \mathcal{T}^{\perp}$, there are no non-zero morphisms from X to $X/t_{\mathcal{T}}X$. But this must mean that $X/t_{\mathcal{T}}X = 0$, so $X = t_{\mathcal{T}}X$, and $X \in \mathcal{T}$.

Starting with a torsion free class $\mathcal{F}$, we see just as easily that the composition of the two maps in the other order is also the identity. They are therefore inverse bijections. It is easy to see that they are order-reversing. □

From the previous theorem, together with Proposition 2.2, the following corollary follows:

Corollary 3.1. *The following pairs of subcategories are the same:*

- $\{(\mathcal{T}, \mathcal{T}^{\perp}) \mid \mathcal{T} \in \mathsf{tors}\, A\}$,
- $\{({}^{\perp}\mathcal{F}, \mathcal{F}) \mid \mathcal{F} \in \mathsf{tf}\, A\}$,
- $\{(\mathcal{X}, \mathcal{Y}) \mid \mathcal{X} = {}^{\perp}\mathcal{Y}, \mathcal{Y} = \mathcal{X}^{\perp})\}$.

4. Posets and lattices

A possible reference for basis material on lattices is [Gr].

A poset is a partially ordered set. In a poset, we say that x covers y if x is greater than y and there is no element z such that $x > z > y$. In this case we write $x \gtrdot y$.

A lattice is a poset in which any two elements x and y have a unique least upper bound (their "join") denoted $x \vee y$, and a unique minimal greatest lower bound (their "meet") denoted $x \wedge y$.

A complete lattice is a lattice such that any subset S of L has a unique least upper bound, which we denote either $\bigvee_{x\in S} x$ or $\bigvee S$, and a unique greatest lower bound, which we denote $\bigwedge_{x\in S} x$ or $\bigwedge S$.

A finite lattice is necessarily complete. The perspective taken in this note is that the desirable infinite generalization of finite lattices are the complete lattices.

A complete lattice necessarily has a minimum element $\hat{0}$ (the meet of all the elements of L) and similarly a maximum element $\hat{1}$.

5. Torsion classes form a complete lattice

The poset $\mathsf{tors}\, A$ clearly has a meet operation given by intersection, since the intersection of two torsion classes again satisfies the defining properties of a torsion class. The same is true for meets of arbitrary collections of torsion classes, for the same reason.

To see the other lattice operation, there are three approaches which all work. Since the left perpendicular/right perpendicular operations are order-reversing bijections between torsion-classes and torsion-free classes, we have that

$$\bigvee_{\mathcal{T}\in S} \mathcal{T} = {}^{\perp}\left(\bigwedge_{\mathcal{T}\in S} \mathcal{T}^{\perp}\right).$$

Since the $\bigwedge$ on the righthand side exists (being given by intersection), so does the $\bigvee$ on the lefthand side.

We can also define the join operation in $\mathsf{tors}\, A$ implicitly. Any poset with a maximum element and a $\bigwedge$ also has a $\bigvee$, which can be defined as follows:

$$\bigvee_{\mathcal{T}\in S} \mathcal{T} = \bigwedge_{\{\mathcal{Y}\in \mathsf{tors}\, A \mid \mathcal{Y}\supseteq \mathcal{T} \forall \mathcal{T}\in S\}} \mathcal{Y}$$

Finally, we can also describe the join explicitly using Proposition 2.1:

$$\bigvee_{\mathcal{T}\in S} \mathcal{T} = T\left(\bigcup_{\mathcal{T}\in S} \mathcal{T}\right)$$

We therefore have the following result:

Proposition 5.1. $\mathsf{tors}\, A$ *is a complete lattice.*

6. Join-irreducible elements in lattices

An element x of a lattice L is called join-irreducible if it cannot be written as the join of two elements both strictly smaller than it, and it is also not the minimum element of the lattice. Especially for finite lattices, the join-irreducible elements can be viewed as "building blocks" of the lattice.

Proposition 6.1. *In a finite lattice L, any element is the join of the join-irreducible elements below it.*

Proof. Suppose $x \in L$ were a minimal counter-example to the statement of the proposition. If x were join-irreducible, it is obviously not a counter-example, so suppose that it is not join-irreducible. We can therefore write $x = y \vee z$ with $y, z < x$. By the assumption that x is a minimal counter-example, y and z can each be written as a join of join-irreducible elements. Joining together these two expressions, we get an expression for x as a join of join-irreducible elements, contradicting our assumption that x was a counter-example. □

The situation for infinite lattices is more complicated. It can still be interesting to consider join-irreducible elements defined as above. However, for our purposes, the following definition is more important. We say that $x \in L$ is completely join irreducible if $\bigvee_{y<x} y < x$. Equivalently, there is an element, which we denote x_* such that $y < x$ if and only if $y \leq x_*$. Note that $\hat{0}$ is not considered to be completely join irreducible. We write $\mathrm{Ji}^c L$ for the completely join irreducible elements of L.

Note that for a finite lattice, $x \in L$ is join irreducible if and only if it is completely join-irreducible. However, this is not true in infinite lattices. For example, consider $[0, 1]$, as an interval in $\mathbb{R}$ with the usual order. Every element except 0 is join-irreducible, but there are no completely join-irreducible elements. This suggests that neither of these notions is necessarily all that useful for general infinite lattices. However, for the lattices we are interested in, the notion of completely join-irreducible elements will turn out to be very important.

Let us return to consider the torsion classes of the Kronecker quiver presented in Example 1.3. Of the torsion classes in the interval isomorphic to a Boolean lattice, the elements covering the minimum are completely join irreducible, while the others are not. Among the other torsion classes, all are completely join-irreducible except the minimum and maximum elements. The unique torsion class which is join-irreducible but not completely join-irreducible is the one composed of all the preinjective modules, labelled $\mathcal{I}$ in the diagram. It is the join of the (infinite) set of torsion classes generated by preinjective modules, but it is not the join of any finite set of torsion classes strictly contained in it.

There are also dual notions of meet irreducible and completely meet irreducible elements of a lattice.

7. Completely join-irreducible torsion classes

Recall that a module B is called a brick if every non-zero endomorphism of B is invertible. A brick is necessarily indecomposable, since projection onto a proper indecomposable summand is a non-invertible endomorphism. Write $\mathsf{br}\, A$ for the A-modules which are bricks.

In the case of the Kronecker quiver, the bricks are the indecomposable modules from the preprojective and preinjective components, together with the quasi-simple module at the bottom of each tube.

In this section, we shall show an important result by Barnard–Carroll–Zhu [BCZ, Theorem 1.5], that there is a bijection between $\mathsf{br}\, A$ and the completely join irreducible elements of $\mathsf{tors}\, A$.

The same result holds for $\mathsf{tf}\, A$, and, by the order-reversing bijection between $\mathsf{tf}\, A$ and $\mathsf{tors}\, A$, the same result also holds for the meet-irreducible elements of $\mathsf{tors}\, A$

and $\mathsf{tf}\, A$. For simplicity, we will focus our attention on $\mathsf{tors}\, A$ and its completely join irreducible elements; everything we prove has analogues in the other settings.

The following lemma says that a torsion class is characterized by the bricks it contains.

Lemma 7.1. *Let $\mathcal{T} \in \mathsf{tors}\, A$. Then*

$$\mathcal{T} = \bigvee_{B \in \mathcal{T} \cap \mathsf{br}\, A} T(B)$$

Proof. Let us write

$$\mathcal{U} = \bigvee_{B \in \mathcal{T} \cap \mathsf{br}\, A} T(B)$$

Clearly, $\mathcal{U} \subseteq \mathcal{T}$. Now suppose that we have some X which is in $\mathcal{T}$ but not in $\mathcal{U}$, and among such X, choose one of minimal dimension. X is clearly not a brick, since otherwise it would be contained in $\mathcal{U}$. Thus it has a non-zero non-invertible endomorphism f. We get a short exact sequence:

$$0 \to f(X) \to X \to X/f(X) \to 0$$

Since $X \in \mathcal{T}$, we have $X/f(X) \in \mathcal{T}$, and since the dimension of $X/f(X)$ is less than that of X, it follows that $X/f(X) \in \mathcal{U}$.

Similarly, though, since $f(X)$ is also a quotient of X, we know $f(X) \in \mathcal{T}$ and thus $f(X) \in \mathcal{U}$. We now see that X is the extension of two objects from $\mathcal{U}$, so it is itself in $\mathcal{U}$, contrary to our assumption. □

We also need the following lemma due to Sota Asai.

Lemma 7.2 ([As, Lemma 1.7(1)]). *If $X \in T(B)$, then either $X \twoheadrightarrow B$ or* $\mathrm{Hom}(X, B) = 0$.

Proof. Suppose that $f \in \mathrm{Hom}(X, B)$ is non-zero. Since X is filtered by quotients of B, we can write $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$, with X_i/X_{i-1} isomorphic to a quotient of B. Consider the smallest i such that $f|_{X_i}$ is non-zero. Since $f|_{X_{i-1}} = 0$, f induces a map from X_i/X_{i-1} to B, and thus from B to B. Since B is a brick, this map must be surjective, so $f|_{X_i}$ is surjective, and thus f is surjective. □

We can now prove the main result of the section:

Theorem 7.1 ([BCZ, Theorem 1.5]). *The map $B \mapsto T(B)$ is a bijection from $\mathsf{br}\, A$ to $\mathrm{Ji}^c\, \mathsf{tors}\, A$.*

Proof. First of all, we want to show that, for B a brick, $T(B)$ is a completely join irreducible torsion class. This requires showing that there is a unique maximum element among all those torsion classes strictly below $T(B)$. We claim that this torsion class can be described as $T(B) \cap {}^{\perp}F(B)$.

Since $B \notin {}^{\perp}F(B)$, it is clear that $T(B) \cap {}^{\perp}F(B)$ is a torsion class strictly contained in $T(B)$. On the other hand, any torsion class strictly contained in $T(B)$ cannot include any module X admitting a surjective map onto B. Thus, by Lemma 7.2, any such torsion class must be contained in ${}^{\perp}B = {}^{\perp}F(B)$. This proves the claim, thus establishing that $T(B)$ is a completely join irreducible torsion class.

On the other hand, by Lemma 7.1, any torsion class can be written as the join of $T(B)$ as B runs through all bricks, so this set exhausts the completely join irreducible torsion classes.

Finally, we want to check that the map from bricks to torsion classes is injective. Suppose that $T(B) = T(B')$, for B and B' two bricks. B' cannot be contained in $T(B)_*$. Thus there is a surjection from B' to B by Lemma 7.2. Reversing the rôles of B' and B, there is also a surjection from B to B'. Therefore B and B' must be isomorphic. □

This theorem is one of the key justifications for the impression that when considering lattices of torsion classes, it is most appropriate to think in terms of the complete versions of lattice-theoretic phenomena. As we saw in the example of the Kronecker quiver, there is a join-irreducible torsion class which is not completely join-irreducible, namely, the additive hull of the preinjective component. In accordance with Theorem 7.1, it does not correspond to any brick in $\mathsf{mod}\, A$. This raises the following interesting question:

Question 7.1. *Is there any way to extend Theorem 7.1 to characterize the join-irreducible but not completely join-irreducible elements of* $\mathsf{tors}\, A$*?*

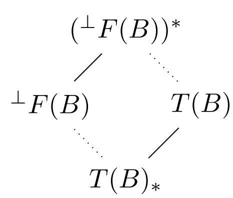
The proof of the following theorem is dual to the proof of Theorem 7.1.

Theorem 7.2. *The map* $B \mapsto F(B)$ *is a bijection from* $\mathsf{br}\, A$ *to* $\mathrm{Ji}^c\, \mathsf{tf}\, A$.

Then, applying Theorem 3.1, we deduce:

Corollary 7.1. *The map* $B \mapsto {}^{\perp}F(B)$ *is a bijection from* $\mathsf{br}\, A$ *to* $\mathrm{Mi}^c\, \mathsf{tors}\, A$.

From Theorem 7.1, Corollary 7.1, and their proofs, we can say that associated to a brick B, there are four torsion classes, arranged as in the following diagram, where the join of the two torsion classes on the middle layer equals the top torsion class, and their meet equals the bottom torsion class.



In the diagram, the edges drawn as undashed lines are cover relations in the lattice of torsion classes. The edges drawn using dashed lines are weak poset relations. In particular, the torsion classes at the endpoints of a dotted line may be equal. Also, the pair of torsion classes not connected by a line are not comparable in the lattice of torsion classes. We follow these conventions in subsequent diagrams.

8. Parenthesis: τ-tilting

We include the following section because it makes the link to another topic of current research related to torsion classes, which was also presented during the spring school.

A torsion class $\mathcal{T}$ is called functorially finite if there is some $X \in \mathsf{mod}\, A$ such that $\mathcal{T} = \mathrm{Gen}(X)$, where $\mathrm{Gen}(X)$ is by definition the collection of quotients of direct sums of copies of X.

In the Kronecker case, which ones are functorially finite? Exactly those not in the Boolean lattice. There is no single module which generates the whole preinjective component and nothing more, and there is no single module which generates any tube without in fact being preprojective (and thus generating all the tubes and more).

Functorially finite torsion classes correspond bijectively to basic support τ-tilting modules; the bijection from basic support τ-tilting modules to torsion classes is Gen.

Functorially finite torsion classes need not form a lattice. There is nothing that guarantees that the intersection of two functorially finite torsion classes will be functorially finite, so in order for them to form a lattice anyway, would have to be a biggest functorially finite torsion class contained in the intersection, and this does not always hold. Generally, for hereditary algebras not of finite type, the functorially finite torsion classes do not form a lattice [IR+, Ri]. Thus, for lattice-theoretic study, it seems preferable not to restrict to functorially finite torsion classes.

9. Semidistributivity

In this section we introduce the notion of semidistributivity of a lattice. See [AN, RST] for more on the subject.

A lattice L is called join semidistributive if $x \vee y = x \vee y'$ implies that $x \vee (y \wedge y')$ is also equal to both of them. It is called completely join semidistributive if given $x \in L$ and a set $S \subseteq L$, such that $x \vee y$ is equal for all $y \in S$, then $x \vee \bigwedge S$ is also equal.

Join semidistributivity and complete join semidistributivity are equivalent for finite lattices. As usual for us, in the infinite setting, the version which we prefer is the complete one.

Complete join semidistributivity is equivalent to saying that, given $x, z \in L$, if we consider $\{y \mid x \vee y = z\}$, then this set, if it is non-empty, has a minimum element. When we say "minimum element," we do not mean only "minimal" (i.e., an element such that there is no element strictly below it), we mean an element which is weakly below all the elements in the set.

Similarly, a lattice is called meet semidistributive if $x \wedge y = x \wedge y'$ implies that $x \wedge (y \vee y')$ is also equal. It is called completely meet semidistributive if given $x \in L$ and a set $S \subseteq L$, such that $x \wedge y$ is equal for all $y \in S$, then $x \wedge \bigvee S$ is also equal. Equivalently, given $x, z \in L$, if we consider $\{y \mid x \wedge y = z\}$, then this set, if non-empty, has a maximum element.

A lattice is called semidistributive if it is join semidistributive and meet semidistributive. It is called completely semidistributive if it is completely join semidistributive and completely meet semidistributive.

Complete semidistributivity is the property which we are going to focus on. We are now going to develop some properties of completely semidistributive lattices.

Proposition 9.1. *In any completely join semidistributive lattice L, every cover $y \gtrdot x$ has a unique completely join irreducible element j such that $x \vee j = y$ and $x \vee j_* = x$.*

Proof. Let $S = \{z \mid x \vee z = y\}$. This set is non-empty, since $y \in S$. Thus, by complete join semidistributivity, it has a minimum element. Call it j.

Any $z < j$ satisfies that $x \vee z < y$, and thus that $x \vee z = x$. It follows that any $z < j$ satisfies that $z \leq x$. Therefore, any $z < j$ satisfies $z \leq x \wedge j$. Since $j \not\leq x$, we have $x \wedge j < j$. Thus every element strictly below j is weakly below $x \wedge j < j$. It follows that j is completely join irreducible, and $j_* = j \wedge x$.

Now suppose that we had some other completely join irreducible element j' such that $x \vee j' = y$ and $x \vee j'_* = x$. Since j is the minimum element of S, we must have $j' > j$. But then $x \geq j'_* \geq j$, which contradicts $x \vee j > x$. Thus j is unique. □

Write $\gamma(y \gtrdot x)$ for the completely join irreducible element defined in the previous proposition.

Similarly, in a completely meet semidistributive lattice L, every cover $y \gtrdot x$ has a unique completely meet irreducible element m such that $m \wedge y = x$ and $m^* \wedge y = y$. Write $\mu(y \gtrdot x)$ for this completely meet irreducible element.

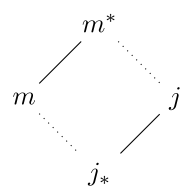
Proposition 9.2. *In a completely semidistributive lattice L, there are inverse bijections κ and κ^d:*

$$\mathrm{Ji}^c(L) \underset{\kappa^d}{\overset{\kappa}{\rightleftarrows}} \mathrm{Mi}^c(L)$$

such that $\kappa(j) = \mu(j \gtrdot j_*)$ and $\kappa^d(m) = \gamma(m^* \gtrdot m)$.

It is standard to call these two maps κ and κ^d but different sources disagree as to which is which.

Proof. Let j be a completely join irreducible element of L, and let $m = \kappa(j) = \mu(j \gtrdot j_*)$. We therefore have the following diagram:

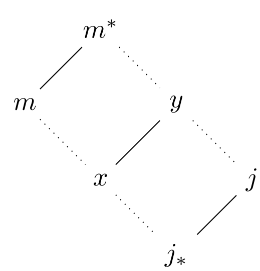


But now it is clear that $\kappa^d(m^* \gtrdot m) = j$, so $\kappa^d \circ \kappa$ is the identity. The dual argument shows that $\kappa \circ \kappa^d$ is the identity, and we have shown that κ and κ^d are inverse bijections. □

We now have the following theorem, which shows that the two labellings of the covers of L differ only by a bijection.

Theorem 9.1. *Let L be a completely semidistributive lattice. Then $\mu(y \gtrdot x) = \kappa(\gamma(y \gtrdot x))$*

Proof. For any $y \gtrdot x$, let $j = \gamma(y \gtrdot x)$ and $m = \mu(y \gtrdot x)$. We therefore have the following diagram, from which the result follows.



□

10. Complete semidistributivity of $\mathsf{tors}\, A$

The fact that lattices of torsion classes are semidistributive was first proved by Garver and McConville [GM]. For not necessarily finite lattices of torsion classes, it turns out to be natural to consider complete semidistributivity.

Theorem 10.1 ([DI+, Theorem 3.1(a)])**.** *$\mathsf{tors}\, A$ is completely semidistributive.*

Proof. We will prove complete meet semidistributivity. Complete join semidistributivity follows from the complete meet semidistributivity of $\mathsf{tf}\, A$, which is established by a dual argument.

Let $\mathcal{X} \in \mathsf{tors}\, A$, and let $S \subseteq \mathsf{tors}\, A$ such that for all $\mathcal{Y} \in \mathsf{tors}\, A$, we have $\mathcal{X} \wedge \mathcal{Y}$ is equal. Let $\mathcal{Z}$ be their common value. Since the meet of torsion classes is intersection, we have that $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y}$ for any $\mathcal{Y} \in S$.

We want to show that $\mathcal{X} \cap \bigvee S = \mathcal{Z}$ also.

Clearly $\mathcal{X} \cap \bigvee S \geq \mathcal{Z}$. To prove the opposite inclusion, let $M \in X \cap \bigvee S$ be a minimal-dimensional counter-example.

Since $M \in \bigvee S$, there is a filtration of M

$$0 = M_0 \subset M_1 \cdots \subset M_r = M$$

with $M_i/M_{i-1} \in \mathcal{Y}_i$, with $\mathcal{Y}_i \in S$.

Consider the short exact sequence:

$$0 \to M_1 \to M \to M/M_1 \to 0$$

Now $M/M_1 \in X \cap \bigvee S$ since M is. Since M_1 is non-zero, the dimension of M/M_1 is less than that of M, and thus by our choice of M, we know that M/M_1 is not a counter-example. Therefore, $M/M_1 \in \mathcal{Z}$, so in particular $M/M_1 \in \mathcal{Y}_1$. On the other hand, we also know that $M_1 \in \mathcal{Y}_1$. Because $\mathcal{Y}_1$ is a torsion class, and therefore closed under extensions, $M \in \mathcal{Y}_1$. We also know $M \in \mathcal{X}$. Therefore $M \in \mathcal{X} \cap \mathcal{Y}_1 = \mathcal{Z}$. This contradicts our choice of M, so it must be that $\mathcal{X} \cap \bigvee S = \mathcal{Z}$. □

11. Consequences of the complete semidistributivity of $\mathsf{tors}\, A$

As we showed in Section 9, a completely semidistributive lattice has a labelling of every cover relation $y \gtrdot x$ by a completely join-irreducible element $\gamma(y \gtrdot x)$, and a labelling of every cover relation by completely meet-irreducible element, $\mu(y \gtrdot x)$ and these two labellings are related by the maps κ and κ^d. We would like to understand what this means in the case of the lattice of torsion classes.

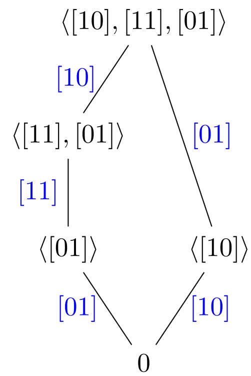
Since we know that the completely join-irreducible torsion classes correspond to bricks by Theorem 7.1, for $\mathcal{Y} \gtrdot \mathcal{X}$ in $\mathsf{tors}\, A$, define $\hat{\gamma}(\mathcal{Y} \gtrdot X) = B$, such that $\gamma(\mathcal{Y} \gtrdot \mathcal{X}) = T(B)$. The following proposition defines $\hat{\gamma}(\mathcal{Y} \gtrdot \mathcal{X})$ directly.

Proposition 11.1. *$\hat{\gamma}(\mathcal{Y} \gtrdot \mathcal{X})$ is the unique brick B which is contained in $\mathcal{Y}$ but not in $\mathcal{X}$.*

Proof. By the complete semidistributivity of $\mathsf{tors}\, A$, we know that there is a unique completely join-irreducible torsion class, $\gamma(\mathcal{Y} \gtrdot \mathcal{X})$, such that $\mathcal{Y} \geq \gamma(\mathcal{Y} \gtrdot \mathcal{X})$ but $\mathcal{X} \not\geq \gamma(\mathcal{Y} \gtrdot \mathcal{X})$. By Theorem 7.1, the completely join-irreducible elements are of the form $T(B)$, for B a brick. We have that $\mathcal{Y} \supseteq T(B)$ and $\mathcal{X} \not\supseteq T(B)$ iff $B \in \mathcal{Y}$ and $B \notin \mathcal{X}$. So there is a unique brick contained in $\mathcal{Y}$ but not in $\mathcal{X}$, and it is $\hat{\gamma}(\mathcal{Y} \gtrdot \mathcal{X})$. □

Dually, $\mu(\mathcal{Y} \gtrdot \mathcal{X}) = {}^{\perp}F(\hat{\gamma}(\mathcal{Y} \gtrdot \mathcal{X}))$.

Example 11.1 (Type A_2)**.** *The brick labelling of the covers of* $\mathsf{tors}\, kQ$ *for* $Q = 1 \leftarrow 2$ *is as follows:*



Example 11.2 (Kronecker quiver)**.** *We revisit the Kronecker quiver from Example 1.3. The brick labels of some of the covers were already shown in Figure 2. Inside the interval that is isomorphic to a Boolean lattice on the set of tubes, one torsion class covers another if they differ exactly in that there is one tube present in one but not the other. In this case the brick labelling the cover relation is the quasi-simple at the bottom of that tube.*

12. Algebra quotients and lattice quotients

A surjective map of lattices $L \twoheadrightarrow L'$ is called a (complete) lattice quotient if it respects the (complete) lattice operations.

Let $\phi : A \twoheadrightarrow A/I$ be an algebra quotient. We can view $\mathsf{mod}\, A/I$ as the subcategory of $\mathsf{mod}\, A$ consisting of modules annihilated by I. We will be interested in the map sending $\mathcal{T}$ in $\mathsf{mod}\, A$ to $\mathcal{T} \cap \mathsf{mod}\, A/I$.

Proposition 12.1. $\mathcal{T} \cap \mathsf{mod}\, A/I$ *is a torsion class for* A/I*.*

Proof. It is easy to check that it satisfies the two defining conditions. □

Proposition 12.2 ([DI+, Proposition 5.7(a)])**.** *If* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair of* $\mathsf{mod}\, A$*, then*

$$(\mathcal{T} \cap \mathsf{mod}\, A/I, \mathcal{F} \cap \mathsf{mod}\, A/I)$$

is a torsion pair of $\mathsf{mod}\, A/I$*.*

Proof. In this proof, when we write $\mathcal{C}^\perp$ or ${}^\perp\mathcal{C}$, we always intend it in the ambient category $\mathsf{mod}\, A$.

Consider $(\mathcal{T} \cap \mathsf{mod}\, A/I)^\perp$. Clearly this contains $\mathcal{F}$. Now suppose we have some module $M \in \mathsf{mod}\, A/I$, $M \notin \mathcal{F}$. There is therefore some $N \in \mathcal{T}$ and some non-zero $f \in \mathrm{Hom}(N, M) \neq 0$. Since $IM = 0$, we must have $f(IN) = 0$, so f descends to a map in $\mathrm{Hom}(N/IN, M)$. But $N/IN \in (\mathcal{T} \cap \mathsf{mod}\, A/I)$. This shows that in fact $M \notin (\mathcal{T} \cap \mathsf{mod}\, A/I)^\perp$. We conclude that the torsion free class in $\mathsf{mod}\, A/I$ which corresponds to $\mathcal{T} \cap \mathsf{mod}\, A/I$ is $\mathcal{F} \cap \mathsf{mod}\, A/I$. □

For $\mathcal{T}$ a torsion class in $\mathsf{mod}\, A$, write $\overline{\phi}(\mathcal{T})$ for $\mathcal{T} \cap \mathsf{mod}\, A/I$.

Proposition 12.3 ([DI+, Proposition 5.7(d)])**.** *If* ϕ *is the quotient* $A \twoheadrightarrow A/I$*, then* $\overline{\phi}$ *is a lattice quotient from* $\mathsf{tors}\, A$ *to* $\mathsf{tors}\, A/I$*.*

Proof. From the definition, it is clear that $\overline{\phi}$ respects the meet operation on $\operatorname{tors} A$. To see that $\overline{\phi}$ respects join, we recall that

$$\bigvee_{\mathcal{T}\in S} \mathcal{T} = {}^{\perp}\left(\bigcap_{\mathcal{T}\in S} \mathcal{T}^{\perp}\right)$$

and the result now follows from Proposition 12.2. □

We are interested in understanding this lattice quotient better. In particular, we will address the question of when two torsion classes in $\operatorname{mod} A$ have the same image under this quotient. For this purpose, we need the following lemma.

Lemma 12.1. *For $\mathcal{U} \leq \mathcal{V}$ in $\operatorname{tors} A$, the following are equivalent:*

(1) $\mathcal{U} < \mathcal{V}$,
(2) $\mathcal{U}^{\perp} \cap \mathcal{V} \neq \{0\}$,
(3) $\mathcal{U}^{\perp} \cap \mathcal{V}$ *contains a brick.*

Proof. The implications (3) implies (2) and (2) implies (1) are obvious.

To see that (1) implies (2), let $X \in \mathcal{V} \setminus \mathcal{U}$. We have a short exact sequence

$$0 \to t_{\mathcal{U}}X \to X \to X/t_{\mathcal{U}}X \to 0.$$

Since $X \notin \mathcal{U}$, we know that $t_{\mathcal{U}}X \neq X$, so $X/t_{\mathcal{U}}X \in {}^{\perp}\mathcal{U}$. On the other hand, $X \in \mathcal{V}$, so $X/t_{\mathcal{U}}X$ is also. Thus $X/t_{\mathcal{U}}X$ witnesses (2).

We now show that (2) implies (3). Suppose that $X \in \mathcal{U}^{\perp} \cap \mathcal{V}$, and suppose that the dimension of X is minimal among non-zero modules in $\mathcal{U}^{\perp} \cap \mathcal{V}$. If X is a brick, we are done, so suppose that X is not a brick. It therefore has a non-invertible non-zero endomorphism f. Let $Y = f(X)$. Now Y is at the same time a quotient and a submodule of X. Since Y is a quotient of X, we know that $Y \in \mathcal{V}$. On the other hand, since Y is a submodule of X, we know that $Y \in \mathcal{U}^{\perp}$. Therefore Y is an element of $\mathcal{U}^{\perp} \cap \mathcal{V}$ of dimension smaller than X, contradicting our choice of X. Thus X must have been a brick. □

From Lemma 12.1, the following proposition is immediate:

Proposition 12.4 ([DI+, Theorem 5.15(b)]). *For $\mathcal{U} \leq \mathcal{V}$ in $\operatorname{tors} A$, $\overline{\phi}(\mathcal{U}) = \overline{\phi}(\mathcal{V})$ if and only if $\mathcal{U}^{\perp} \cap \mathcal{V}$ contains no modules annihilated by I, or equivalently contains no bricks annihilated by I.*

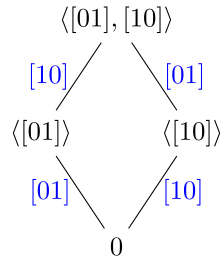
Another way to formulate the proposition is that if $\mathcal{U} \leq \mathcal{V}$, then $\overline{\phi}(\mathcal{U}) \neq \overline{\phi}(\mathcal{V})$ precisely if there is some module in $\mathcal{U}^{\perp} \cap \mathcal{V}$ which is annihilated by I.

Also, we have the following proposition. We write $\hat{\gamma}_A$ and $\hat{\gamma}_{A/I}$ for the labellings associated to covers in $\operatorname{tors} A$ and $\operatorname{tors} A/I$, respectively.

Proposition 12.5 ([DI+, Theorem 5.15(a)]). *If $\mathcal{Y} \gtrdot \mathcal{X}$ in $\operatorname{tors} A$ and $\overline{\phi}(\mathcal{Y}) \gtrdot \overline{\phi}(\mathcal{X})$ in $\operatorname{tors} A/I$, then $\hat{\gamma}_A(\mathcal{Y} \gtrdot \mathcal{X}) = \hat{\gamma}_{A/I}(\overline{\phi}(\mathcal{Y}) \gtrdot \overline{\phi}(\mathcal{X}))$.*

Proof. If $\mathcal{Y} \gtrdot \mathcal{X}$ in $\operatorname{tors} A$, then there is a unique brick from $\operatorname{mod} A$ in $\mathcal{X}^{\perp} \cap \mathcal{Y}$, namely $\hat{\gamma}_A(\mathcal{Y} \gtrdot \mathcal{X})$. Given that $\overline{\phi}(\mathcal{Y}) \neq \overline{\phi}(\mathcal{X})$, this brick must in fact lie in $\operatorname{mod} A/I$. It is therefore the unique brick in $\overline{\phi}(\mathcal{X})^{\perp} \cap \overline{\phi}(\mathcal{Y})$, and thus equals $\hat{\gamma}_{A/I}(\mathcal{Y} \gtrdot \mathcal{X})$. □

Example 12.1 (Type A_2). *Let $A = kQ$, where $Q = 1 \leftarrow 2$. Let I be the ideal of A generated by the arrow. A/I is the path algebra of two vertices and no arrows; $\operatorname{tors} \operatorname{mod} A/I$ is as follows:*



We see that it is obtained from the lattice $\mathsf{tors}\, A$ *by identifying the two torsion classes* $\langle [01], [11] \rangle$ *and* $\langle [01] \rangle$*, which differ only in modules which are not in* $\mathsf{mod}\, A/I$*. We further see that the labels of the cover relations which remain cover relations in* $\mathsf{tors}\, A/I$ *receive the same labels as cover relations in* $\mathsf{tors}\, A$ *and as cover relations in* $\mathsf{tors}\, A/I$*, consistent with Proposition 12.5.*

In the next section, we will see how to combine Proposition 12.4 with the labelling $\hat{\gamma}$. In order to do that, we need another important structural result about $\mathsf{tors}\, A$.

13. $\mathsf{tors}\, A$ IS WEAKLY ATOMIC

A lattice is called weakly atomic if in any interval $[u, v]$ with $u < v$, there is some pair of elements x, y with $x \lessdot y$. (This property is referred to as arrow-separatedness in the current version of [DI+] and as cover-separatedness in the current version of [RST], but they will be updated to reflect the standard terminology.) The interval $[0, 1]$ in $\mathbb{R}$, with the usual order, is an example of a lattice which is not weakly atomic (since it has no cover relations at all).

In this section, we will prove the following two theorems.

Theorem 13.1 ([DI+])**.** $\mathsf{tors}\, A$ *is weakly atomic.*

Theorem 13.2 ([DI+])**.** *Let* $\phi : A \twoheadrightarrow A/I$ *be an algebra quotient. For* $\mathcal{U} \subseteq \mathcal{V}$*, we have that* $\overline{\phi}(\mathcal{U}) = \overline{\phi}(\mathcal{V})$ *iff all covers in* $[\mathcal{U}, \mathcal{V}]$ *are labelled by bricks which are not annihilated by* I*.*

On the way to proving these theorems, we first prove the following proposition, which can be viewed as a relative version of Theorem 7.1.

Proposition 13.1 ([DI+, Theorem 3.4])**.** *Let* $\mathcal{U} \leq \mathcal{V}$ *be two torsion classes. The map* $B \mapsto T(B) \vee \mathcal{U}$ *is a bijection from* $\mathsf{br}(\mathcal{U}^\perp \cap \mathcal{V})$ *to* $\mathrm{Ji}^c[\mathcal{U}, \mathcal{V}]$*.*

Proof. Let $B \in \mathsf{br}(\mathcal{U}^\perp \cap \mathcal{V})$. Let $\mathcal{Y} = \mathcal{U} \vee T(B)$. Also consider the torsion class $\mathcal{X} = \mathcal{Y} \cap {}^\perp F(B)$. Since $B \in \mathcal{U}^\perp$, we have that ${}^\perp F(B) \supseteq \mathcal{U}$, so $\mathcal{X}$ also lies in $[\mathcal{U}, \mathcal{V}]$. Now $\mathcal{X}$ is strictly contained in $\mathcal{Y}$ since it does not contain B. But any torsion class containing $\mathcal{U}$ which is strictly contained in $\mathcal{Y}$ cannot include any module admitting a surjective map onto B. By Lemma 7.2, any such torsion class is therefore contained in ${}^\perp B = {}^\perp F(B)$. This shows that $\mathcal{Y}$ is completely join-irreducible in $[\mathcal{U}, \mathcal{V}]$ and that $\mathcal{Y}$ covers $\mathcal{X}$.

We now show that all the completely join irreducible elements of $[\mathcal{U}, \mathcal{V}]$ correspond to some brick as above. Any torsion class can be written as the join of the torsion classes corresponding to the bricks it contains, so any torsion class in $[\mathcal{U}, \mathcal{V}]$ can be written as the join of $\mathcal{U}$ and a set of torsion classes of the form $T(B)$ for B

lying in some subset of $\mathcal{U}^\perp \cap \mathcal{V}$. It follows that the only completely join irreducible elements of $[\mathcal{U}, \mathcal{V}]$ are those of the form $\mathcal{U} \vee T(B)$.

Finally, the map from bricks to torsion classes is invertible. If $\mathcal{Y}$ is a completely join irreducible torsion class in $[\mathcal{U}, \mathcal{V}]$, with $\mathcal{X}$ the unique torsion class in $[\mathcal{U}, \mathcal{V}]$ which it covers, then the brick corresponding to $\mathcal{Y}$ is $\hat{\gamma}(\mathcal{Y} \gtrdot \mathcal{X})$. □

Based on this, we can now easily establish the following proposition:

Proposition 13.2 ([DI+])**.** *Let $\mathcal{U} < \mathcal{V}$ be two torsion classes in* $\mathsf{mod}\, A$*. Then there are covers in $[\mathcal{U}, \mathcal{V}]$ labelled by each brick in $\mathcal{U}^\perp \cap \mathcal{V}$, and no others.*

Note that $\mathcal{U}^\perp \cap \mathcal{V}$ is non-empty by Lemma 12.1.

Proof. It is clear that no other brick can appear as a label since if $\mathcal{V} \geq \mathcal{Y} \gtrdot \mathcal{X} \geq \mathcal{U}$, and $\hat{\gamma}(\mathcal{Y} \gtrdot \mathcal{X}) = B$ then $B \in \mathcal{Y} \subseteq \mathcal{V}$ and $B \in \mathcal{X}^\perp \subseteq \mathcal{U}^\perp$.

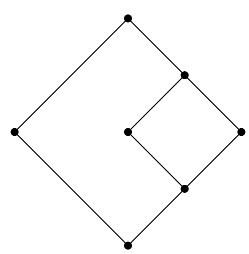
For the converse direction, if B is a brick in $\mathcal{U}^\perp \cap \mathcal{V}$, then by Proposition 13.1, there is a completely join irreducible in $[\mathcal{U}, \mathcal{V}]$ corresponding to B, and the cover relation down from it in $[\mathcal{U}, \mathcal{V}]$ is labelled by B. □

Theorem 13.1 follows directly from Proposition 13.2, since if $\mathcal{U} < \mathcal{V}$, then by Lemma 12.1, $\mathcal{U}^\perp \cap \mathcal{V}$ contains a brick.

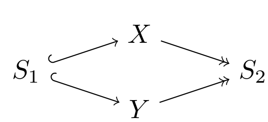
Theorem 13.2 follows as well, by combining Proposition 12.4 with Proposition 13.2.

14. A combinatorial application: finite semi-distributive lattices

Consider the following lattice:



This lattice is semidistributive. Suppose that it were isomorphic to $\mathsf{tors}\, A$ for some A. We see that this lattice has four (completely) join-irreducible elements and four (completely) meet-irreducible elements, so $\mathsf{mod}\, A$ would necessarily have four bricks by Theorem 7.1. We see that two of the bricks would have to be simple, call them S_1 and S_2, and there would be maps as follows, with X and Y being the other two bricks:



There is no such module category. Results of [AP], extending [Ja], can also be used to construct many examples of finite semidistributive lattices which are not lattices of torsion classes.

In light of this, it would seem unlikely that representation theory could help us to understand general finite semidistributive lattices. Nonetheless, it turns out that it can: there is a kind of combinatorial relaxation of the notion of torsion class which allows us to construct exactly all finite semidistributive lattices. This is the main result of [RST], and I shall not attempt to prove it here, but I will at least state the result.

Given any finite set, which we will call Ш, and a reflexive relation $\to$ on Ш, for a subset $\mathcal{C} \subset$ Ш, we can define $\mathcal{C}^\perp = \{Y \in Ш \mid \forall X \in \mathcal{C}, X \not\to Y\}$, and ${}^\perp\mathcal{C} = \{X \mid \forall Y \in \mathcal{C}, X \not\to Y\}$. Torsion pairs in Ш are then defined to be pairs of subsets $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T}^\perp = \mathcal{F}$ and $\mathcal{T} = {}^\perp\mathcal{F}$. We can then define $\mathsf{tors}(Ш, \to)$ to be the set of pairs $(\mathcal{T}, \mathcal{F})$, ordered by inclusion on $\mathcal{T}$. If we allow ourselves to start with any set Ш and reflexive relation $\to$, this construction is so general as to be able to construct any finite lattice, as was discovered by Markowsky [Ma].

Our inspiration for the above construction (though not Markowsky's original inspiration) obviously comes from thinking of Ш as the set of bricks of some module category, and $\to$ as the relation "there exists a non-zero morphism". It turns out that by putting additional conditions on $\to$, we can restrict ourselves to obtaining exactly the finite semidistributive lattices; these relations come from demanding that $\to$ look more like existence of a non-zero morphism, in a sense which I will now describe.

Starting from a reflexive relation $\to$, define two other relations, $\twoheadrightarrow$ and $\hookrightarrow$. We define $X \twoheadrightarrow Y$ iff whenever $Y \to Z$ then $X \to Z$. Similarly, we define $X \hookrightarrow Y$ iff whenever $Y \to Z$ then $X \to Z$. Again, the intuition from representation theory is clear: if M and N are A-modules and there is a surjection from M to N then whenever there is a non-zero map from N to some L, then there is also a non-zero map from M to L and dually for injections. (Note, though, that if we take Ш $= \mathsf{br}\,\mathsf{mod}\,A$ and take $\to$ to be "there exists a non-zero morphism", the relations $\twoheadrightarrow$ and $\hookrightarrow$ defined as above are not exactly "there exists a surjection" and "there exists an injection". See [RST, Section 8] for more details.)

We say that a reflexive relation $\to$ on Ш is factorizable if it satisfies the following two conditions:

- For any $X, Z \in$ Ш with $X \to Z$, there exists $Y \in$ Ш such that
$$X \twoheadrightarrow Y \hookrightarrow Z.$$
- Any of $X \twoheadrightarrow Y \twoheadrightarrow X$ or $X \hookrightarrow Y \twoheadrightarrow X$, or $X \hookrightarrow Y \hookrightarrow X$ imply $X = Y$.

As is probably clear, the motivating intuition for the first condition is that a non-zero morphism can be factored as a surjection followed by an injection.

We can now state the main result of [RST]:

Theorem 14.1 ([RST, Theorem 1.2])**.** *Let* Ш *be a finite set, and* $\to$ *a reflexive factorizable relation on* Ш*. Them* $\mathsf{tors}(Ш, \to)$ *is a semidistributive lattice, and every semidistributive lattice arises in this way for a choice of* Ш *and* $\to$ *which is unique up to isomorphism.*

I close with the following question:

Question 14.1. *Is there a way to interpret any finite semidistributive lattice as the lattice of torsion classes of a "real" category?*

The question is deliberately worded somewhat imprecisely. Another way to ask the question would be to ask for a representation-theoretic meaning to the construction of finite semidistributive lattices of [RST].

Acknowledgements

I would like to thank my coauthors on [IR+, DI+, RST], from whom I have learned a great deal. It is my pleasure to acknowledge NSERC and the Canada Research Chairs program for their financial support. Thanks to Nathan Reading and Alexander Garver for helpful comments on this manuscript. I am extremely grateful to have been given the opportunity to present this material at Zhejiang University in 2018 and at the Isfahan School on Representations of Algebras in 2019. Thanks to Fang Li for the invitation to Zhejiang and to the organizers of the Isfahan School and Conference on Representations of Algebras, and in particular Javad Asadollahi, for the invitation to speak in Isfahan, and for the invitation to prepare this contribution to the special issue.

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